Leaving Certificate Examination 2007 – Mathematics – Higher Level Paper 2 – Solutions

1. (a)

$x = 5 + 7\cos\theta \qquad\qquad y = 7\sin\theta$

$\Rightarrow \quad \cos\theta = \dfrac{x-5}{7} \qquad \Rightarrow \quad \sin\theta = \dfrac{y}{7}$

$\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad \dfrac{(x-5)^2}{49} + \dfrac{y^2}{49} = 1$

$\Rightarrow \quad \boxed{(x-5)^2 + y^2 = 49}$

(b)

$x^2 + y^2 - 4x - 6y + 5 = 0 \qquad\qquad x^2 + y^2 - 6x - 8y + 23 = 0$

$c_1(2,3) \qquad\qquad c_2(3,4)$

$r_1 = \sqrt{4+9-5} = \sqrt{8} = 2\sqrt{2} \qquad\qquad r_2 = \sqrt{9+16-23} = \sqrt{2}$

(i) Touch internally if $d_{c_1 \to c_2} = r_1 - r_2$

$d_{c_1 \to c_2} = \sqrt{(3-2)^2 + (4-3)^2} = \sqrt{2}$

$r_1 - r_2 = 2\sqrt{2} - \sqrt{2} = \sqrt{2} = d_{c_1 \to c_2}$

$\Rightarrow$ Circles touch internally.

(ii) Since $r_1 = 2r_2$ we simply reflect c_1 by central symmetry through c_2

$(2,3) \to (3,4) \to (4,5)$

$\Rightarrow$ Point of contact is $(4,5)$.

(c) Horizontal distance from centre to y-axis is 3.

Vertical distance from this point on the y-axis to the circumference $= \dfrac{8}{2} = 4$.

We can therefore use Pythagoras' Theorem to deduce that $r = 5$.

Centre must therefore be $(3,5)$

Equation: $(x-h)^2 + (y-k)^2 = r^2$

$\Rightarrow \quad \boxed{(x-3)^2 + (y-5)^2 = 25}$

2. (a) $\vec{x} = -2\vec{i} + 5\vec{j}$ $\qquad \overrightarrow{xy} = -6\vec{i} - 8\vec{j}$

$\Rightarrow \quad \vec{y} - \vec{x} = -6\vec{i} - 8\vec{j}$

$\Rightarrow \quad \vec{y} = -6\vec{i} - 8\vec{j} + \vec{x}$

$\Rightarrow \quad \vec{y} = -6\vec{i} - 8\vec{j} - 2\vec{i} + 5\vec{j}$

$\Rightarrow \quad \vec{y} = -8\vec{i} - 3\vec{j}$

(b) $\vec{a} = 5\vec{i}$ $\qquad \vec{b} = \sqrt{3}\,\vec{i} + 3\vec{j}$

(i) $\overrightarrow{ab} = \vec{b} - \vec{a} = \left(\sqrt{3} - 5\right)\vec{i} + 3\vec{j}$

$\overrightarrow{ab} \cdot \vec{b} = \left[\left(\sqrt{3} - 5\right)\vec{i} + 3\vec{j}\right] \cdot \left[\sqrt{3}\,\vec{i} + 3\vec{j}\right]$

$= \sqrt{3}\left(\sqrt{3} - 5\right) + 9 = 3 - 3\sqrt{5} + 9 = 12 - 3\sqrt{5} \neq 0$

$\Rightarrow \quad \overrightarrow{ab} \not\perp \vec{b}$

(ii) $\vec{c} = k\vec{b}$

$\Rightarrow \quad \vec{c} = k\sqrt{3}\,\vec{i} + 3k\,\vec{j}$

$\overrightarrow{ac} = \vec{c} - \vec{a} = \left(k\sqrt{3} - 5\right)\vec{i} + 3k\,\vec{j}$

$\overrightarrow{ac} \perp \vec{b} \Rightarrow \overrightarrow{ac} \cdot \vec{b} = 0$

$\Rightarrow \quad \left[\left(k\sqrt{3} - 5\right)\vec{i} + 3k\vec{j}\right] \cdot \left[\sqrt{3}\,\vec{i} + 3\vec{j}\right] = 0$

$\Rightarrow \quad \sqrt{3}\left(k\sqrt{3} - 5\right) + 9k = 0$

$\Rightarrow \quad 3k - 5\sqrt{3} + 9k = 0 \Rightarrow \; 12k = 5\sqrt{3} \Rightarrow \boxed{k = \frac{5\sqrt{3}}{12}}$

(c) $\vec{p} = 3\vec{i} + 4\vec{j}$ $\qquad \vec{q} = 5\vec{i} + 12\vec{j}$

(i) $\vec{r} = \frac{65t}{16}\left(\frac{\vec{p}}{|\vec{p}|} + \frac{\vec{q}}{|\vec{q}|}\right) = \frac{65t}{16}\left(\frac{3\vec{i} + 4\vec{j}}{5} + \frac{5\vec{i} + 12\vec{j}}{13}\right)$

$= \frac{\cancel{65}t}{16}\left(\frac{13\left(3\vec{i} + 4\vec{j}\right) + 5\left(5\vec{i} + 12\vec{j}\right)}{\cancel{65}}\right) = \frac{t}{16}\left(39\vec{i} + 52\vec{j} + 25\vec{i} + 60\vec{j}\right)$

$= \frac{t}{16}\left(64\vec{i} + 112\vec{j}\right) = t\left(4\vec{i} + 7\vec{j}\right) = 4t\vec{i} + 7t\vec{j}$

(ii) $\vec{p} \cdot \vec{r} = \left(3\vec{i} + 4\vec{j}\right) \cdot \left(4t\vec{i} + 7t\vec{j}\right) = 12t + 28t = 40t$

$\vec{q} \cdot \vec{r} = \left(5\vec{i} + 12\vec{j}\right) \cdot \left(4t\vec{i} + 7t\vec{j}\right) = 20t + 84t = 104t$

(iii) $\vec{r}$ is on the bisector of $\angle poq$ if $\angle por = \angle qor$

$\cos|\angle por| = \frac{\vec{p} \cdot \vec{r}}{|\vec{p}||\vec{r}|} = \frac{40t}{(5)\left(\sqrt{16t^2 + 49t^2}\right)} = \frac{40\cancel{t}}{5\cancel{t}\sqrt{65}} = \frac{8}{\sqrt{65}}$

$\cos|\angle qor| = \frac{\vec{q} \cdot \vec{r}}{|\vec{q}||\vec{r}|} = \frac{104t}{(13)\left(\sqrt{16t^2 + 49t^2}\right)} = \frac{104\cancel{t}}{13\cancel{t}\sqrt{65}} = \frac{8}{\sqrt{65}}$

$\cos|\angle por| = \cos|\angle qor| \Rightarrow |\angle por| = |\angle qor| \Rightarrow r$ is on bisector of $\angle poq$.

3. (a)

$$(1,1) \to (0,0)$$
$$(8,-5) \to (7,-6) = (x_1, y_1)$$
$$(5,-2) \to (4,-3) = (x_2, y_2)$$
$$A = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}|(7)(-3)-(4)(-6)| = \frac{1}{2}|-21+24| = \frac{3}{2}$$

(b) $f:(x,y) \to (x', y,)$ where $x' = 4x+2y$ and $y' = -3x-y$

$K: x+y=0$

(i) $x' = 4x+2y$ $\qquad$ $y' = -3x-y$

$\Rightarrow \quad y = -y'-3x$

$\Rightarrow \quad x' = 4x+2(-y'-3x)$

$\Rightarrow \quad x' = 4x-2y'-6x$

$\Rightarrow \quad 2x = -x'-2y'$

$\Rightarrow \quad x = \dfrac{-x'-2y'}{2}$ $\qquad \Rightarrow \quad y = -y'+\dfrac{3x'+6y'}{2}$

$\Rightarrow \quad y = \dfrac{3x'-4y'}{2}$

Now, $K: \quad x+y=0$

$\Rightarrow \quad \dfrac{-x'-2y'}{2}+\dfrac{3x+4y}{2}=0$

$\Rightarrow \quad -x'-2y'+3x'+4y'=0$

$\Rightarrow \quad 2x'+2y'=0$

$\Rightarrow \quad x'+y'=0 \qquad \Rightarrow \quad K$ is its own image under f.

(ii) $p=(1,-1) \Rightarrow f(p)=(2,-2)$

$q=(3,-3) \Rightarrow f(q)=(6,-6)$

$$\frac{|pq|}{|f(p)\,f(q)|} = \frac{\sqrt{(3-1)^2+(-3+1)^2}}{\sqrt{(6-2)^2+(-6+2)^2}} = \frac{\sqrt{8}}{\sqrt{32}} = \frac{2\sqrt{2}}{4\sqrt{2}} = \frac{1}{2}$$

$\Rightarrow \quad |pq| : |f(p)\,f(q)| = 1:2$

(c) (i) $k(3x-5y+6)+l(5x-7y+4)=0$

$\Rightarrow$ $(3k+5l)x-(5k+7l)y+6k+4l=0$

This represents the equation of a line as it is in the form $ax+by+c=0$.

Now, Let (x_1, y_1) be the point of intersection of the two lines

$L_1: 3x-5y+6=0$ and $L_2: 5x-7y+4=0$

$(x_1, y_1)\in L_1 \Rightarrow 3x_1-5y_1+6=0$

$(x_1, y_1)\in L_2 \Rightarrow 5x_1-7y_1+6=0$

Now, substitute (x_1, y_1) into $k(3x-5y+6)+l(5x-7y+4)=0$

$\Rightarrow$ $k(3x_1-5y_1+6)+l(5x_1-7y_1+4)=0$

$\Rightarrow$ $k(0)+l(0)=0$ $\Rightarrow$ $0=0$ $\Rightarrow (x_1, y_1)$ satisfies the equation

$\Rightarrow$ $k(3x-5y+6)+l(5x-7y+4)=0$ represents any line through (x_1, y_1) the point of intersection of L_1 and L_2.

(ii) $(3k+5l)x-(5k+7l)y+6k+4l=0$ has a slope of 2

$\Rightarrow$ $\dfrac{3k+5l}{5k+7l}=2 \Rightarrow 3k+5l=10k+14l \Rightarrow 7k=-9l \Rightarrow \boxed{k=-\dfrac{9l}{7}}$

(iii) If $k=1$ then the line becomes $3x-5y+6+l(5x-7y+4)=0$.

This represents every line through the point of intersection of L_1 and L_2 except for $5x-7y+4=0$. This is because, given that the k value cannot now be zero, we can't have a situation where L_1 disappears from the equation.

4. (a) $\left(\cos A+\sin A\right)^2$

$=\cos^2 A+2\sin A\cos A+\sin^2 A$

$=\underbrace{\cos^2 A+\sin^2 A}_{=1}+\underbrace{2\sin A\cos A}_{=\sin 2A}$

$=1+\sin 2A$ Q.E.D.

(b) $6\cos^2 x+\sin x-5=0$

$\Rightarrow$ $6\left(1-\sin^2 x\right)+\sin x-5=0$

$\Rightarrow$ $6-6\sin^2 x+\sin x-5=0$

$\Rightarrow$ $6\sin^2 x-\sin x-1=0$ $\ldots$let $y=\sin x$

$\Rightarrow$ $6y^2-y-1=0$

$\Rightarrow$ $(3y+1)(2y-1)=0$

$\Rightarrow$ $y=-\dfrac{1}{3}$ $y=\dfrac{1}{2}$

$\Rightarrow$ $\sin x=-\dfrac{1}{3}$ $\sin x=\dfrac{1}{2}$

$\Rightarrow$ $x=\{199°,341°\}$ $x=\{30°,150°\}$

$\Rightarrow$ $x=\{30°,150°,199°,341°\}$

(c) (i) $|oa|=|oc|=r\Rightarrow$ isosceles triangle $\Rightarrow|\angle cao|\Rightarrow|\angle aco|=\alpha$

$\Rightarrow$ $|\angle aoc|=180-2\alpha$

Using the cosine rule

$|ac|^2=r^2+r^2-2r^2\cos(180-2\alpha)$

$\Rightarrow$ $|ac|^2=2r^2+2r^2\left(\cos 180\cos 2\alpha+\sin 180\sin 2\alpha\right)$

$\Rightarrow$ $|ac|^2=2r^2+2r^2\left(-\cos 2\alpha\right)$

$\Rightarrow$ $|ac|^2=2r^2\left(1-\cos 2\alpha\right)$

$\Rightarrow$ $|ac|^2=2r^2\left(2\sin^2\alpha\right)$

$\Rightarrow$ $|ac|^2=4r^2\sin^2\alpha$

$\Rightarrow$ $|ac|=2r\sin\alpha$

(ii) $\dfrac{\text{Area of sector } boc+\text{Area of }\triangle aoc}{\text{Area of semi-circle}}=\dfrac{1}{2}$

$\Rightarrow$ $\dfrac{\frac{1}{2}r^2(2\alpha)+\frac{1}{2}r(2r\sin\alpha)\sin\alpha}{\pi r^2/2}=\dfrac{1}{2}$

$\Rightarrow$ $\dfrac{2r^2\alpha+2r^2\sin^2\alpha}{\pi r^2}=\dfrac{1}{2}$

$\Rightarrow$ $\dfrac{2\cancel{r^2}\left(\alpha+\sin^2\alpha\right)}{\pi\cancel{r^2}}=\dfrac{1}{2}$

$\Rightarrow$ $2\alpha+2\sin^2\alpha=\dfrac{\pi}{2}$

5. (a) $$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2\cancel{x}}{3\cancel{x}}=\frac{2}{3}$$

(b) $$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos(A+(-B))=\cos A\cos(-B)-\sin A\sin(-B)$$

$$(\cos(-B)=\cos B \text{ and } \sin(-B)=-\sin B)$$

$$\Rightarrow \quad \cos(A-B)=\cos A\cos B-\sin A(-\sin B)=\cos A\cos B+\sin A\sin B$$

Now, $\sin A=\cos\left(\frac{\pi}{2}-A\right)$ and $\cos A=\sin\left(\frac{\pi}{2}-A\right)$

Thus $$\sin(A+B)=\cos\left(\frac{\pi}{2}-(A+B)\right)$$

$$=\cos\left(\left(\frac{\pi}{2}-A\right)-B\right)$$

$$=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$$

$$=\sin A\cos B+\cos A\sin B$$ Q.E.D.

(c) In Δprs, $$\frac{|pr|}{\sin 30}=\frac{h}{\sin 60}$$

$$\Rightarrow \quad |pr|=\frac{h\sin 30}{\sin 60}=\frac{h\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{h}{\sqrt{3}}$$

In Δqrs, $$\frac{|qr|}{\sin 60}=\frac{h}{\sin 30}$$

$$\Rightarrow \quad |qr|=\frac{h\sin 60}{\sin 30}=\frac{h\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}}=h\sqrt{3}$$

Also, $$3c^2=13h^2$$

$$\Rightarrow \quad c=h\sqrt{\frac{13}{3}}$$

Using Cosine Rule,

$$|pq|^2=|pr|^2+|qr|^2-2|pr||qr|\cos|\angle prq|$$

$$\Rightarrow \quad \cos|\angle prq|=\frac{|pr|^2+|qr|^2-|pq|^2}{2|pr||qr|}=\frac{\frac{h^2}{3}+3h^2-\frac{13h^2}{9}}{2\left(\frac{h}{\cancel{\sqrt{3}}}\right)\left(h\cancel{\sqrt{3}}\right)}=\frac{\frac{h^2}{3}+3h^2-\frac{13h^2}{9}}{2h^2}$$

$$\Rightarrow \quad \cos|\angle prq|=\frac{3+27-13}{18}=\frac{17}{18}$$

$$\Rightarrow \quad |\angle prq|=19\cdot 188^\circ$$

6. (a) (i) $6! = 720$

(ii) $5! = 120$

(b) If $u_n = l\alpha^n + m\beta^n$ then $u_{n+1} = l\alpha^{n+1} + m\beta^{n+1}$ and $u_{n+2} = l\alpha^{n+1} + m\beta^{n+2}$. Then

$$pu_{n+2} + qu_{n+1} + ru_n$$
$$= p\left(l\alpha^{n+2} + m\beta^{n+2}\right) + q\left(l\alpha^{n+1} + m\beta^{n+2}\right) + r\left(l\alpha^n + m\beta^n\right)$$
$$= l\left(p\alpha^{n+2} + q\alpha^{n+1} + r\alpha^n\right) + m\left(p\beta^{n+2} + q\beta^{n+1} + r\beta^n\right)$$
$$= l\alpha^n\left(p\alpha^2 + q\alpha + r\right) + m\beta^n\left(p\beta^2 + q\beta + r\right)$$
$$= l\alpha^n(0) + m\beta^n(0)$$
$= 0$ as α and β are roots of the equation $px^2 + qx + r = 0$.

(c) (i) $p = \left(\dfrac{r}{r+w}\right)\left(\dfrac{r-1}{r+w-1}\right) = \dfrac{r(r-1)}{(r+w)(r+w-1)}$

(ii) When $w = 1$, $p = \dfrac{1}{2}$

$\Rightarrow$ $\dfrac{\cancel{r}(r-1)}{\cancel{r}(r+1)} = \dfrac{1}{2}$

$\Rightarrow$ $2r - 2 = r + 1$

$\Rightarrow$ $\boxed{r = 3}$

(iv) Let $w = 2$, $p = \dfrac{1}{2}$

$\dfrac{r(r-1)}{(r+2)(r+1)} = \dfrac{1}{2} \quad \Rightarrow \quad \dfrac{r^2 - r}{r^2 + 3r + 2} = \dfrac{1}{2}$

$\Rightarrow$ $2r^2 - 2r = r^2 + 3r + 2$

$\Rightarrow$ $r^2 - 5r - 2 = 0$ …no integer solution.

Let $w = 4$, $p = \dfrac{1}{2}$

$\dfrac{r(r-1)}{(r+3)(r+2)} = \dfrac{1}{2} \quad \Rightarrow \quad \dfrac{r^2 - r}{r^2 + 5r + 6} = \dfrac{1}{2}$

$\Rightarrow$ $2r^2 - 2r = r^2 + 5r + 6$

$\Rightarrow$ $r^2 - 7r - 6 = 0$ …no integer solution.

Let $w = 6$, $p = \dfrac{1}{2}$

$\dfrac{r(r-1)}{(r+6)(r+5)} = \dfrac{1}{2} \quad \Rightarrow \quad \dfrac{r^2 - r}{r^2 + 11r + 30} = \dfrac{1}{2}$

$\Rightarrow$ $2r^2 - 2r = r^2 + 11r + 30$

$\Rightarrow$ $r^2 - 13r - 30 = 0$

$\Rightarrow$ $(r-15)(r+2) = 0$

$\Rightarrow$ $\boxed{r = 15}$, $r = -2$

$\Rightarrow$ $w = 6$, $r = 15$, $p = \dfrac{1}{2}$

7. (a) (i) $7 \times 6 \times 5 \times 4 = 840$

(ii) Number of selections with no vowel $= 4 \times 3 \times 2 \times 1 = 24$

$\Rightarrow$ Number of selections with at least on vowel $= 840 - 24 = 816$

(b) (i) Let A be the event of getting two identical numbers.

Let B be the event of getting a total of five.

$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$ but in this case $P(A \cap B) = 0$ as the two events are independent.

$\Rightarrow$ $P(A \text{ or } B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$

(ii) $P(A \times B \text{ is at least } 2(A+B)) = 1 - P(A \times B \text{ is less than } 2(A+B))$

$= 1 - \frac{25}{36} = \frac{11}{36}$

(c) (i) $S_7 = \frac{7}{2}\{2a + 6d\} = 7\{a + 3d\}$

$\text{Mean} = \frac{S_7}{7} = a + 3d$

(ii) $\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$

$(x_1 - \overline{x})^2 = (a - (a + 3d))^2 = 9d^2$

$(x_2 - \overline{x})^2 = (a + d - (a + 3d))^2 = 4d^2$

$(x_3 - \overline{x})^2 = (a + 2d - (a + 3d))^2 = d^2$

$(x_4 - \overline{x})^2 = (a + 3d - (a + 3d))^2 = 0$

$(x_5 - \overline{x})^2 = (a + 4d - (a + 3d))^2 = d^2$

$(x_6 - \overline{x})^2 = (a + 5d - (a + 3d))^2 = 4d^2$

$(x_7 - \overline{x})^2 = (a + 6d - (a + 3d))^2 = 9d^2$

$\Rightarrow$ $\sigma = \sqrt{\frac{9d^2 + 4d^2 + d^2 + 0 + d^2 + 4d^2 + 9d^2}{7}} = \sqrt{\frac{28d^2}{7}} = \sqrt{4d^2} = 2d$

Q.E.D.

8. (a) $p+q=1 \Rightarrow q=1-p \Rightarrow pq=p(1-p)=p-p^2$

$$\frac{d}{dp}(pq)=1-2p=0 \Rightarrow \boxed{p=\frac{1}{2}}$$

(b) (i)

$f(x)=(1+x)^m \quad \Rightarrow f(0)=1$

$f'(x)=m(1+x)^{m-1} \quad \Rightarrow f'(0)=m$

$f''(x)=m(m-1)(1+x)^{m-2} \quad \Rightarrow f''(0)=m(m-1)$

$f'''(x)=m(m-1)(m-2)(1+x)^{m-3} \quad \Rightarrow f'''(0)=m(m-1)(m-2)$

$f(x)=f(0)+f'(0)x+f''(0)x^2+f'''(0)x^3+\cdots$

$\Rightarrow \quad f(x)=1+mx+m(m-1)x^2+m(m-1)(m-2)x^3+\cdots$

(ii) $$u_r=\frac{m(m-1)(m-2)\text{..............}(m-r+1)}{r!}x^r$$

Let

$$R=\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right|=$$

$$\lim_{r\to\infty}\left|\frac{m(m-1)(m-2)\text{.....}(m-r+1)(m-r)x^{r+1}}{(r+1)!}\times\frac{r!}{m(m-1)(m-2)\text{.....}(m-r+1)x^r}\right|$$

$$=\lim_{r\to\infty}\left|\frac{m-r}{r+1}\right||x|=\lim_{r\to\infty}\left|\frac{\frac{m}{r}-1}{1+\frac{1}{r}}\right||x|=|x||-1|=|x|$$

Thus the series is convergent for $|x|<1$, i.e. $-1<x<1$.

(c) $I=\int_0^1 \tan^{-1}x\,dx \qquad$ Let $\quad u=\tan^{-1}x \qquad dv=dx$

$$du=\frac{1}{1+x^2}dx \qquad v=x$$

$\Rightarrow \quad I=\left[x\tan^{-1}x\right]_0^1-\int_0^1\frac{x}{1+x^2}dx \qquad u=1+x^2$

$$du=2xdx \Rightarrow xdx=\frac{1}{2}du$$

$$x=0\Rightarrow u=1,\ x=1\Rightarrow u=2$$

$\Rightarrow \quad I=\left[x\tan^{-1}x\right]_0^1-\frac{1}{2}[\ln u]_1^2=\left[\frac{\pi}{4}\right]-\frac{1}{2}[\ln 2-\ln 1]=\frac{\pi}{4}-\frac{1}{2}\ln 2=\boxed{\frac{\pi}{4}-\ln\sqrt{2}}$